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## ABSTRACT

In this paper, we extend the toll pricing framework previously developed for traffic networks to ones that include several modes of transportation such as walking, driving, and using public conveyance (e.g., buses, subways, and trains). To determine tolls, we construct a user equilibrium and system optimal model. In both models, we assume that users adopt strategies or hyperpaths to travel between each origin-destination pair and the demand between each pair is fixed. However, the choice between driving and using public transportation is determined by a binomial logit function. As in the case of traffic networks, the set of valid tolls can be obtained from the solution to the system problem and the equilibrium conditions for the user problem. Then, objective functions similar to those for traffic networks can be used to determine an optimal toll vector. We provide a numerical example to illustrate our approach.



**Key words**: Congestion pricing, Hyperpaths, Strategies, Traffic equilibrium, Transit networks.



## RÉSUMÉ

Dans cet article, nous présentons une extension du cadre d'analyse des péages pour inclure plusieurs modes de transport, tels que le transport en camion, la voiture privée et le mode piéton. Afin de déterminer les péages, nous formulons un problème d'équilibre égoïste et un problème normatif. Dans les deux modèles, nous supposons que les usagers adoptent des stratégies (ou hyperchemins) pour leur choix de route. Le choix entre la voiture privée et le transport en commun est déterminé par une fonction logit binomiale. Un ensemble de péages valide est obtenu à partir de la solution du modèle normatif et les conditions d'équilibre égoïste. Ensuite, des fonctions objectifs similaires à celles utilisées pour le trafic de voitures privées sont utilisées pour déterminer les péages optimaux. Un exemple numérique est également présenté.



**Mots clés**: Hyperchemins, Stratégies, Équilibre de trafic, Réseau de transport en commun, péages.

# 1 Introduction

In urban transportation systems, congestion pricing employs market forces to allocate limited road and highway capacities among travellers by their need to travel and their willingness to pay. The concept involves charging higher prices for usage during peak travel periods. When willing to pay higher prices, travellers can travel with reduced congestion during peak periods, using the modes, and along the routes of their choosing. Although there are many successful congestion pricing projects implemented in both the U.S. and other countries, Hecker [14] testified to the Joint Economic Committee, United States Congress, that these projects are not always able to demonstrate benefits for the full range of users. Those who are financially able to use tolled highway lanes may see a reduction in travel time. On the other hand, there is little system wide reduction in travel times in some cases and congestion in other cases may increase on alternative routes or outside tolled areas. In addition, congestion pricing in and of itself is often thought of as a factor that encourages ridesharing and use of transit. While this is true for Singapore's Electronic Road Pricing (formerly Central Area Licensing Scheme), the shift to alternative modes in the United States is close to non-existence in some cases. (See [6].)

In transportation science, the literature on congestion pricing focuses on systems with a single mode of transportation–automobiles (see, e.g., [1], [12], [15] and [17]). However, to estimate the effects of congestion pricing on other forms of transportation requires models that address systems with multiple modes (see, e.g., [5], [9] and [19]). Our goal is to develop such congestion pricing models and demonstrate, e.g., their usefulness in estimating the impact on the transit ridership due to toll pricing and, perhaps, transit fare adjustments.

In this paper, the framework for congestion pricing is similar to the one in [4] and [12]. The principal idea is that a set of tolls or a toll vector is considered *valid* if it forces the solution to the tolled user equilibrium problem to be the same as that of the system optimal (or, more simply, system) problem. For systems with multiple modes of transportation, existing user equilibrium (or user) models (e.g., [5], [8] and [3]) assume that each traveller adopt a single path to reach his or her final station when using the transit system. However, such an approach may be unnecessarily simplistic. Instead, our models assume as in [18] and [21] that travellers employ *strategies* or *hyperpaths* when using the transit system. At each transit station, a strategy or hyperpath specifies a subset of transit lines that travellers can use to reach their destinations. Transit lines in this subset are called *attractive* (see, e.g., [21]) and it is assumed that travellers board the first attractive line to arrive while waiting at a

station.

In addition to the use of strategies, we also assume that the travel demand between each origin-destination (OD) pair is fixed and use the binomial logit function as described in [8] and [3] to determine the share of travellers who will use the different modes of transportation for each OD pair. Along with the calibrating parameters, each logit function depends on the travellers' perception of the generalized travel cost of every mode of transportation between the OD pair. If all parameters are equal, transportation modes with less travel cost should have larger shares of travellers.

For the remainder, Section 2 describes components of models with multiple modes of transportation such as the underlying networks for various travel options, strategy or hyperpath assumption, node and arc probabilities, and feasible solutions in terms of hyperpath and link flow variables. Section 3 formulates a system problem that maximizes the partial net user benefit and Section 4 demonstrates that the user equilibrium conditions are equivalent to a variational inequality. Section 5 shows that tolls on arcs corresponding to walking and embarking or disembarking from transit vehicles can be set to zero and proposes several toll pricing problems. A numerical example illustrating potential uses of these toll pricing problems is described in Section 6. Finally, Section 7 summarizes our results.

## 2 Problem Description

To formulate the multi-modal version of user and system problems, we consider two broad categories of travel options–*automobile* (or *auto*) and *mixed* options. In the auto option, travellers use private automobiles exclusively to travel from origins to destinations. For the mixed option, travellers either walk, bicycle, or drive a short distance to nearby stations, use public transportation (e.g., buses, subways, metros, and commuter trains) to arrive at some final stations, and walk from there to their destinations. For simplicity, the discussion below only mentions the following two mixed options explicitly.

**Walk-Metro:** Travellers walk to metro stations, use metro lines to reach the final stations, and walk from there to their destinations.

**Auto-Metro:** Travellers drive to metro stations, use metro lines to reach the final stations, and walk from there to their destinations.

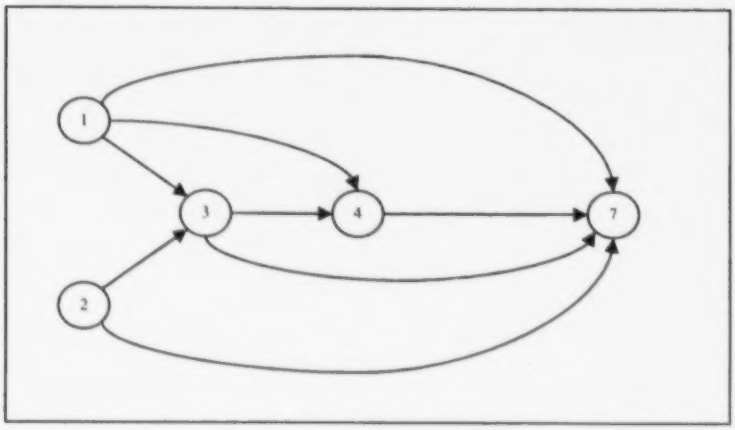



Figure 1: An underlying network for the auto option

## 2.1 Underlying Network Structure

For the auto option, we let $G^a(N^o \cup N^a \cup N^{am}, E^a)$ be the auto network (or graph) encompassing all possible routes using roads and highways from origins to destinations and nearby metro stations. In this network, $N^o$, $N^a$, and $N^{am}$ are the sets of nodes representing origins and destinations, road intersections, and combinations of road intersections and metro stations, respectively. For the latter (i.e., the set $N^{am}$), it is often convenient to consider a road intersection and a nearby metro station as a single node. Then, $E^a$ is the set of arcs (or, more descriptively, *auto arcs*) corresponding to roads and highways connecting the nodes. For example, Figure 1 displays an auto network in which $N^o = \{1, 2, 7\}$, $N^a = \emptyset$, and $N^{am} = \{3, 4\}$ and $E^a$ is the set of arcs shown in the figure.

Similarly, $G^{wm}(N^o \cup N^{am} \cup N^m \cup N^{tp}, E^w \cup E^m)$ represents the underlying network for the walk-metro option. The first two node sets, $N^o$ and $N^{am}$, are as defined above. Nodes in $N^m$ and $N^{tp}$ denote pure metro stations and transfer points, respectively. Nguyen and Pallottino [18] refer to nodes in the set $N^{am} \cup N^m$ as *stop nodes* and we refer to them as *station nodes*. Arcs in $E^w$ (or walk arcs) correspond to walking from origins to metro stations and from metro stations to destinations. On the other hand, $E^m$ consists of segments of metro lines connecting two transfer points and embarking and disembarking arcs between metro stations and transfer points. Figure 2 display an example of a walk-metro network where, as before, $N^o = \{1, 2, 7\}$ and $N^{am} = \{3, 4\}$. In addition to these node sets, $N^m = \{5, 6\}$ and $N^{tp} = \{11, 12, \ldots, 19\}$, $E^w = \{(1, 3), (2, 3), (6, 7)\}$, and $E^m$ consists of the remaining arcs.

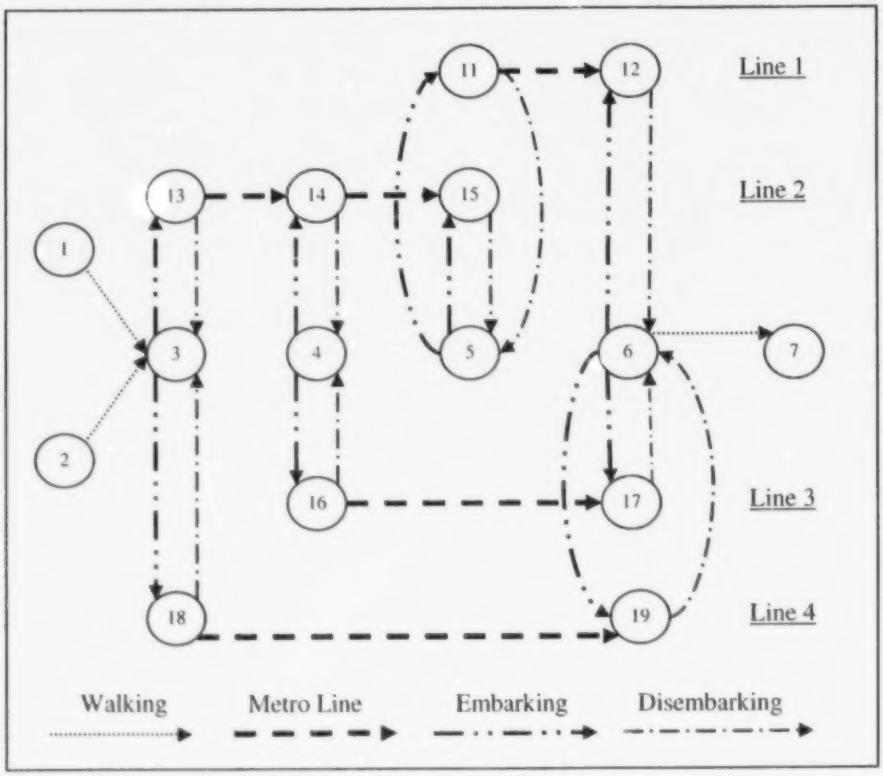



Figure 2: An underlying network for the walk-metro option

To illustrate, a path $1 \rightarrow 3 \rightarrow 13 \rightarrow 14 \rightarrow 15 \rightarrow 5 \rightarrow 11 \rightarrow 12 \rightarrow 6 \rightarrow 7$ corresponds to walking from an origin at node 1 to the metro station at node 3, boarding or embarking on metro line 2 at transfer point 13, disembarking from line 2 via transfer point 15 to metro station at node 5, embarking on metro line 1 at transfer point 11, disembarking from line 1 via transfer point 12 to metro station 6, and walking from there to the destination at node 7. Although it is also possible to add walking arcs, e.g., between metro stations 4 and 5, we assume that the distance between any pair metro stations is too far for walking to simplify our presentation.

Lastly, $G^{am}(N^o \cup N^{am} \cup N^m \cup N^{tp}, E^m \cup \hat{E}^a \cup \hat{E}^w)$ denotes the underlying network for the auto-metro option. The node sets and $E^m$ are as defined earlier. For the remaining arc sets, $\hat{E}^a = \{(i,j) : i \in N^o, j \in N^{am}\} \subseteq E^a$ and $\hat{E}^w = \{(i,j) : i \in N^{am}, j \in N^o\} \subseteq E^w$, where $E^a$ and $E^w$ are as defined earlier. For example, Figure 3 displays an auto-metro network. Except for $\hat{E}^a = \{(1,3),(2,3)\}$, and $\hat{E}^w = \{(6,7)\}$, all the node and arc sets are as defined earlier. (To simplify our presentation, our

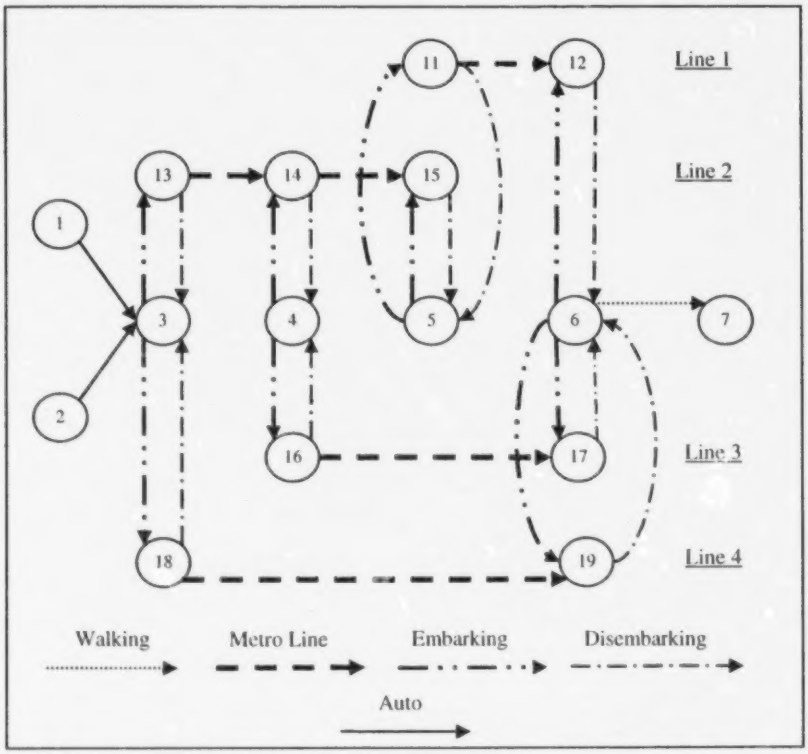



Figure 3: An underlying network for the auto-metro option

definition of the auto-metro network implicitly assumes that travellers drive from an origin $i$ to a metro station $j \in N^{am}$ along a single arc in the set $\hat{E}^a$. However, it is easy to relax this assumption by adding the set $N^a$ to the node set of $G^{am}$ and let $\hat{E}^a$ consists of auto arcs belonging paths from origins to metro stations instead.)

The main difference between the networks in Figures 2 and 3 are arcs (1, 3) and (2, 3). In Figure 2, these arcs correspond to walking. However, the ones in Figure 3 represent driving instead.

In most cases, the auto-metro and walk-metro networks such as those in Figures 2 and 3 are rather large and appear unwieldy. It is often easier to understand and simpler to illustrate when we consolidate a station node along with its adjacent transfer points into one. To illustrate, the network in Figure 4 is a consolidated version of the one in Figure 2. For example, node 3 in Figure 4 is a consolidation of station node 3 and transfer points 13 and 18 in Figure 2.

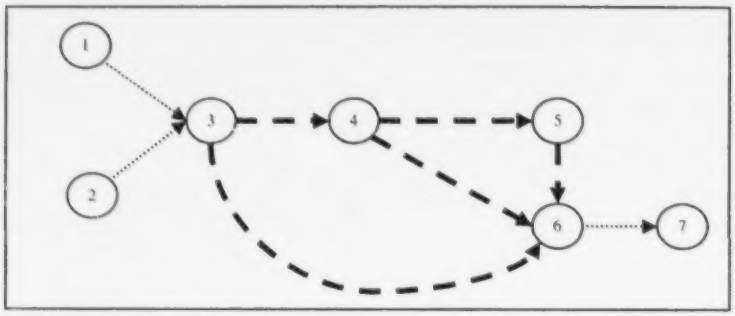



Figure 4: A consolidated version of the walk-metro network in Figure 2

## 2.2 Strategies and Hyperpaths

When travelling from origins to destinations, we assume that travellers use strategies. Travellers specify their strategies by providing at each node in the underlying network a set of immediate successor nodes (or successors) that allow them to reach their destinations. In [21], Spiess and Florian call successor nodes in this set *attractive*. The conditions under which a strategy is valid are different for different underlying networks. For a given OD pair $(o, d)$, a strategy is *valid* for a mixed network (walk-metro or auto-metro) if the following hold:

a. The set of attractive successors at node $d$ is empty.

b. The set of attractive successors at node $o$ contains exactly one element. By our construction, this element must be a metro station, i.e., an element of the set $N^{am} \cup N^{m}$.

c. At each non-station nodes, the number of attractive successors is at most one. (There is no restriction on the number of attractive successors at any station node.)

d. The set of attractive successors at every node induces an acyclic subnetwork with the following properties:

   i. Node $o$ has no predecessor.

   ii. For every node $i \notin \{o, d\}$ in the subnetwork, there is a directed path from $o$ to $d$ via node $i$.

Nguyen and Pallottino [18] refer to the induced subnetwork in condition (d) as a *hyperpath*. To illustrate, consider the walk-metro network in Figure 2. Table 1 provides a valid strategy for OD pair (1, 7) and Figure 5 displays the subnetwork or hyperpath induced by the strategy in Table 1. (Nodes

| Node | Attractive Successors | Node | Attractive Successors |
|---|---|---|---|
| 1 | 3 | 11 | $\emptyset$ |
| 2 | $\emptyset$ | 12 | $\emptyset$ |
| 3 | 13, 18 | 13 | 14 |
| 4 | 16 | 14 | 4 |
| 5 | $\emptyset$ | 15 | $\emptyset$ |
| 6 | 7 | 16 | 17 |
| 7 | $\emptyset$ | 17 | 6 |
| | | 18 | 19 |
| | | 19 | 6 |

Table 1: A strategy for the walk-metro network in Figure 2

with no successors are not shown in Figure 5.) Note that this hyperpath consists of two simple paths from node 1 to node 7. One path reaches node 7 via node 13 and the other does it via node 18.

For the auto network, a strategy is *valid* if, in addition to conditions (a), (b), and (d) specified above, the number of attractive successors at every node is at most one. Under these conditions, a hyperpath for the auto network contains exactly one directed path for each OD pair. This is the same assumption used in traffic assignment models in the literature (see, e.g., [10]).

### 2.3 Hyperpaths and Link Flows

For an OD pair $(o, d)$, let $G_h(N^h, E^h)$ be a subnetwork induced by a hyperpath $h$. We associate with each arc $(i, j) \in E^h$ an access probability $\pi_{ij}^h \in [0, 1]$, a probability that the traveller will use arc $(i, j)$ when arrived at node $i$. (It is our convention to let $\pi_{ij}^h = 0$ for all arc $(i, j) \notin E^h$.) A vector of access (or arc) probabilities, $\pi^h$, is meaningful for hyperpath $h$ if $\sum_{j \in E^h(i^+)} \pi_{ij}^h = 1$ for all $i \in N^h$, where $E^h(i^+)$ is the set of attractive successors for node $i$. When $G_h(N^h, E^h)$ is a subnetwork of the auto network, then it must correspond to a simple directed path and $\pi_{ij}^h = 1, \forall (i, j) \in E^h$. For subnetworks corresponding to hyperpaths in a mixed network, $\pi_{ij}^h$ may be fractional if node $i$ is a station node, i.e., $i \in N^{am} \cup N^m$. When $i$ is a non-station node, $E^h(i^+)$ must be a singleton set, e.g., $E^h(i^+) = \{j\}$, and $\pi_{ij}^h$ must equal one.

In a mixed network, the access probabilities at a station node can be determined, for example,

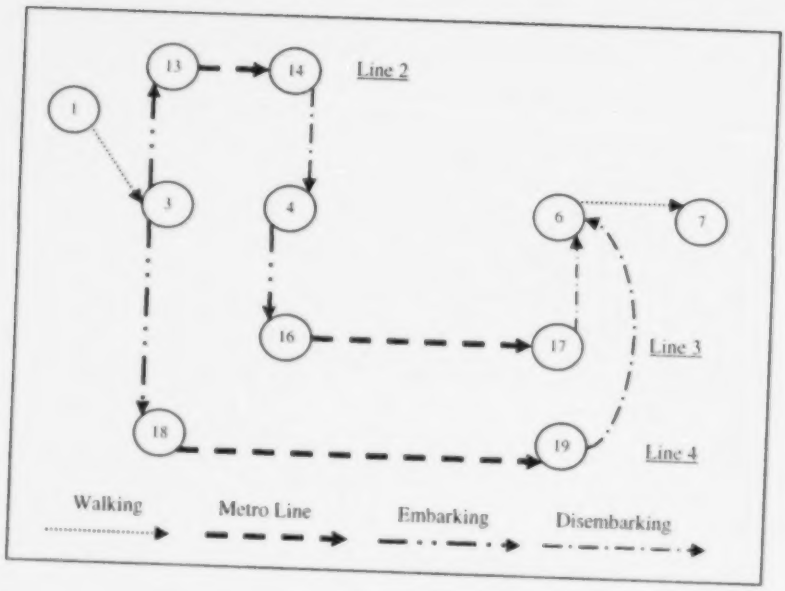



Figure 5: The subnetwork or hyperpath induced by the strategy in Table 1

from the frequencies of the metro lines. Consider a station node $i \in N^h$. Then, every node $j \in E^h(i^+)$ must be a transfer point. Assume that the inter-arrival time of the (metro) trains serving transfer point $j$ is distributed independently and exponentially with rate or frequency $f_j$ (e.g., $f_j$ is the number of trains per hour). Under these assumptions, the access probability for arc $(i, j)$ is

$$\pi_{ij}^h = \frac{f_j}{\sum_{k \in E^h(i^+)} f_k}.$$

The above expression is the probability that the train serving transfer point $j$ arrives before trains serving transfer point $k \in E^h(i^+)$, where $k \neq i$. For the hyperpath in Figure 5, assume that the frequencies of lines 2, and 4 are 10 and 20. Then, $\pi_{3,13}^h = \frac{10}{(10+20)} = \frac{1}{3}$ and $\pi_{3,18}^h = \frac{2}{3}$. For the origin node, the cardinality of $E^h(o^+)$ is one and the arc $(o, j)$, where $j \in E^h(o^+)$ must be a walk or auto arc. Thus, $\pi_{oj} = 1$. Equivalently, the "frequency" of a walk or auto arc is infinite.

For a mixed network, this is also a waiting time associated with every station node in a hyperpath. Using the frequencies of metro lines and the properties of exponential distribution (see, e.g., [20]), the expected waiting time for the first attractive train to arrive at station node $i$ is

$$w_i^h = \frac{1}{\sum_{k \in E^h(i^+)} f_k}.$$

Continuing with the above example, the waiting at node 3 is $\frac{1}{30}$ of an hour, i.e., 2 minutes. Because of the infinite frequency interpretation, there is no waiting at the origin (node 1) and the immediate predecessors (node 6) of the destination. In general, the waiting time at a non-station node in a mixed network or at any node in the auto network is zero.

Given an access probability vector $\pi^h$, it is also possible to compute the probabilities associated with each node in the subnetwork $G_h(N^h, E^h)$. In particular, we compute the node probabilities in a topological order as follows:

Step 1: Set the node probability for the origin to one, i.e., $\tau_o^h = 1$.

Step 2: Select node $i \in N^h$ in a topological order and set $\tau_j^h = \sum_{i \in E^h(j^-)} \tau_i^h \pi_{ij}^h$, where $E^h(j^-) = \{i : (i,j) \in E^h\}$ or set of immediate predecessors for node $j$.

For hyperpaths in the auto network, the above calculations imply that every node in the hyperpath has a probability of one. For a mixed network, Figure 6 gives an example of node probabilities using a consolidated representation of a walk-metro network. If we assign 24 units of flow to the hyperpath in the figure, we expect $\frac{7}{24} * 24$ or 7 units of flow to pass through node 6.

From the node probabilities, the expected waiting time associated with hyperpath $h$ is $w^h = \sum_{i \in E^h} \tau_i^h w_i^h$. In addition, the combination of node and arc probabilities also allows the amount of flow on each arc to be determined from the flow assigned to each hyperpath. Let $x_h$ denote the flow assigned to hyperpath $h$. Then, $\tau_j^h x_h$ is the amount of flow expected to pass through node $j \in E^h$ and $\pi_{ij}^h \tau_i^h x_h$ is the proportion of the flow that traverses arc $(i,j)$. Thus, the total amount of flow on an arc $(i,j) \in E^a \cup E^w \cup E^m$ is $v_{ij} = \sum_{h \in P} \tau_i^h \pi_{ij}^h x_h$, where $P = \bigcup_k P_k$ and $P_k$ is the set of all hyperpaths for OD pair $k$.

## 2.4 Feasible Region

To define the feasible region of the multi-modal problems, let $b_k$ denote the (fixed) travel demand of OD pair $k$ and further decompose the path set $P_k$ into $P_k^a$, $P_k^{am}$, and $P_k^{wm}$, each denoting the set of hyperpaths in the auto, auto-metro, and walk-metro networks, respectively, i.e., $P_k = P_k^a \cup P_k^{am} \cup P_k^{wm}$.

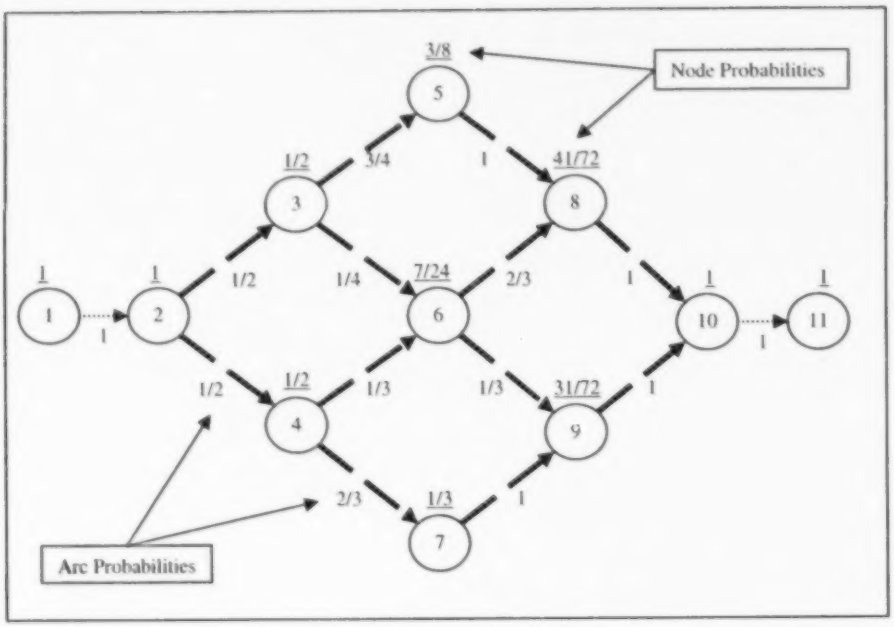



Figure 6: Node and arc probabilities of a hyperpath for OD pair (1,11) in a consolidated walk-metro network

Then, a feasible aggregate (link) flow vector, $v$, must satisfy the following linear system:

$$
\begin{aligned}
v_{ij} &= \sum_k \frac{1}{\gamma_k} \left( \sum_{h \in P_k} \tau_i^h \pi_{ij}^h x_h \right), && \forall (i,j) \in E^a \\
v_{ij} &= \sum_k \left( \sum_{h \in P_k} \tau_i^h \pi_{ij}^h x_h \right), && \forall (i,j) \in E^w \cup E^m \\
\sum_{h \in P_k^{am} \cup P_k^{wm}} x_h &= t_k, && \forall k \\
\sum_{h \in P_k^a} x_h &= b_k - t_k, && \forall k \\
x_h &\geq 0, && \forall h \in \bigcup_k P_k
\end{aligned}
$$

where $\gamma_k$ represents the occupancy rate for automobiles travelling between OD pair $k$. The first two sets of equations calculate the link flows from the hyperpath flow, $x_h$. By construction, $\tau_i^h = 1$ and $\pi_{ij}^h \in \{0,1\}$ for all auto arcs in hyperpath $h$, i.e., for all $(i,j) \in E^a \cap E^h$. Thus, $\tau_i^h \pi_{ij}^k$ acts as an indicator whether arc $(i,j)$ is in hyperpath $h$. The factor $\frac{1}{\gamma_k}$ in the first set of equation also converts the number of travellers into the number of automobiles on arc $(i,j)$. So, $v_{ij}$ in the first set represents the number of automobiles and $v_{ij}$ in the second denotes the number of travellers instead. In addition, the first

two sets of equations also indicate how models in subsequent sections account for flows in different networks. The index of the second summation in these two sets of equations is over all hyperpaths for OD pair $k$, i.e., hyperpaths in both auto and mixed networks. For the first set of equations, $(i, j)$ is an auto arc and can appear in both auto and auto-metro networks. Thus, $v_{ij}$ consists for flows on hyperpaths that include arc $(i, j)$ in both networks. The similar holds for the walk and metro arcs in the second set of equations.

In the third and fourth set of equations, $t_k$ represents the share of $b_k$ that uses the mixed travel options. These two sets of equations ensure that the amount of flow assigned to each hyperpath sums to the travel demand for OD pair $k$. The last set of inequalities forces the flow on each hyperpath to be nonnegative. Note that there is no constraint on $t_k$. However, the nonnegativity of $x_h$ ensures that each $t_k \in [0, b_k]$. In matrix form, the above system is equivalent to the following:

$$\begin{aligned} v &= Z \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ A_1 x_1 &= t \\ A_2 x_2 &= b - t \\ x_1, x_2 &\geq 0. \end{aligned}$$

In the first constraint, $v$ is a vector of arc flows, $Z$ is a matrix that is similar in structure to the arc-hyperpath incidence matrix, $x_1$ and $x_2$ are vectors whose components correspond to flows on hyperpaths in $P_k^{am} \cup P_k^{wm}$ and $P_k^a$, respectively. For the next two constraints, $A_1$ and $A_2$ are OD pair-hyperpath incidence matrices and $b$ and $t$ are vectors of travel demands and the corresponding shares that use the mixed options. When convenient, we also let $x$ denote a vector with $x_1$ and $x_2$ as its components and $X$ denote the set of all $(v, x, t)$ feasible to the above system.

## 3 System Optimum Problem

In the literature, a typical objective of a system problem with fixed demands is to minimize the total delay. In the multi-modal case with fixed demands, it also makes sense to do the same. However, to design a tolling scheme that encourages the use of public transportation, it is natural to choose a solution that maximizes the use of public transportation among those with minimum total delay. Unfortunately, an optimal solution to a system problem is typically unique. As an alternative, one

approach is to maximize a partial net user benefit, i.e., the benefit associated with making trips under the mixed options minus the total delay associated with waiting, walking and travelling in the metro system or automobiles. The expression of waiting time is $\sum_h w^h x_h$ where, as defined previously, $w^h$ is the waiting time associated with hyperpath $h$. To compute travel delay, let $s(v)$ be a vector function. If $(i,j) \in E^w \cup E^m$, the element $s_{ij}(v_{ij})$ of $s(v)$ represents the travel time for each traveler. For arc $(i,j) \in E^a$, $s_{ij}(v_{ij})$ represents the travel time for each vehicle. Then, the total travel delay $s(v)^T v$ has two components, one is the delay on the auto network, $\sum_{(i,j)\in E^a} s_{ij}(v_{ij})v_{ij}$ and the other is the delay on the mixed networks, $\sum_{(i,j)\in E^w \cup E^m} s_{ij}(v_{ij})v_{ij}$.

To determine the benefit associated with making trips using the mixed options, we first assume as in [8] that the choice between the two travel options, auto or mixed, depends on binomial logit functions, $t_k^a(\rho_k^a, \rho_k^m)$ and $t_k^m(\rho_k^a, \rho_k^m)$, which specifies the demands for the auto and mixed options, respectively. Both $t_k^a(\rho_k^a, \rho_k^m)$ and $t_k^m(\rho_k^a, \rho_k^m)$ are functions of the (generalized) travel costs, $\rho_k^a$ and $\rho_k^m$, under the auto and mixed travel options, respectively. Then,

$$t_k^a(\rho_k^a, \rho_k^m) = b_k \frac{\exp\{-\lambda^a - \eta\rho_k^a\}}{\exp\{-\lambda^a - \eta\rho_k^a\} + \exp\{-\lambda^m - \eta\rho_k^m\}} = \frac{b_k}{1 + \exp\{-(\lambda^m - \lambda^a) - \eta(\rho_k^m - \rho_k^a)\}}$$

$$t_k^m(\rho_k^a, \rho_k^m) = b_k \frac{\exp\{-\mu^m - \eta\rho_k^m\}}{\exp\{-\lambda^a - \eta\rho_k^a\} + \exp\{-\lambda^m - \eta\rho_k^m\}} = \frac{b_k}{1 + \exp\{(\lambda^m - \lambda^a) + \eta(\rho_k^m - \rho_k^a)\}}$$

where $\lambda^a$, $\lambda^m$, and $\eta$ are parameters for calibration and, as defined previously, $b_k$ denote the fixed travel demand for OD pair $k$. Note that $t_k^a(\rho_k^a, \rho_k^m) + t_k^m(\rho_k^a, \rho_k^m) = b_k$ and it is only necessary to consider one of these two functions explicitly in the system problem. The expression for the travel demand under the mixed option, $t_k^m(\rho_k^a, \rho_k^m)$, can be equivalently written as function of the difference between the two travel costs, i.e., $u_k = \rho_k^m - \rho_k^a$, as follows:

$$t_k^m(u_k) = \frac{b_k}{1 + \exp\{(\lambda^m - \lambda^a) + \eta u_k\}}.$$

(The superscript $m$ on $t_k$ is superfluous and we remove it from the expressions below.) In terms of $u_k$, the inverse demand function for the mixed option is

$$u_k(t_k) = \frac{1}{\eta}\left[\ln\left(\frac{b_k - t_k}{t_k}\right) - (\lambda^m - \lambda^a)\right]$$

and, as in the single mode case (see, e.g.,[10]), the benefit associated with making $t_k$ using the mixed option can be calculated from the above inverse demand function as follows:

$$\int_0^{t_k} u_k(z)dz.$$

Instead of maximizing the net user benefit, the system problem below minimizes the negative of the partial net user benefit instead.

$$\begin{aligned} \text{SOPT}: \quad \min \quad & s(v)^T v + W^T x - \sum_k \int_o^{t_k} u_k(z)dz \\ \text{s.t.} \quad & v = Z \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ & A_1 x_1 = t \\ & A_2 x_2 = b - t \\ & x_1, x_2 \geq 0 \end{aligned}$$

where $W$ is the vector of waiting times on hyperpaths, $w_h$. It is easy to show that $\int_0^{t_k} -u_k(z)dz$ is convex when $u_k(z)$ is the inverse demand function derived from the binomial logit functions shown above. When $s(v)^T v$ is also convex, the following theorem provides optimality conditions for SOPT.

**Theorem 1** *Assume that $s(v)^T v$ is convex. A feasible point $(v^*, x^*, t^*)$ is optimal to SOPT if and only if there exist $\rho^1$ and $\rho^2$ such that the following hold:*

$$\begin{aligned} Z^T(s(v^*) + \nabla s(v^*)^T v^*) + W &\geq \begin{bmatrix} A_1^T \rho^1 \\ A_2^T \rho^2 \end{bmatrix} \\ u(t^*) &= \rho^1 - \rho^2 \\ [s(v^*) + \nabla s(v^*)^T v^*]^T v^* + W^T x^* &= (t^*)^T \rho^1 + (b - t^*)^T \rho^2 \end{aligned}$$

**Proof:** The result follows from the KKT conditions for SOPT and is similar to the one in [12]. ■

The last optimality condition in Theorem 1 has an intuitive interpretation. The left hand side of the equation, $[s(v^*) + \nabla s(v^*)^T v^*]^T v^* + W^T x^*$, is an expression for the travel cost based on private (travel and waiting times) and marginal social costs, $\nabla s(v^*)^T v^*$. (See, e.g., [1].) The right hand side is an expression for the travel cost based on the KKT multipliers, $\rho^1$ and $\rho^2$. In this case, $\rho_k^1$ is the generalized travel cost for OD pair $k$ using the mixed options and the associated delay is $\sum_k t_k^* \rho_k^1$ or $(t^*)^T \rho^1$. Similarly, the second term, $(b - t^*)^T \rho^2$, is the generalized cost associated with the auto option. At optimality, the last condition ensures that travel costs based on these two methods of calculation must be the same.

## 4 User Equilibrium

For a feasible solution $(v, x, t)$, the cost to each traveller using hyperpath $h \in P = \bigcup_k P_k$ is as follows:

$$C_h(v,x,t) = \sum_{(i,j)\in E^m \cup E^w} \tau_i^h \pi_{ij}^h s_{ij}(v) + \sum_{(i,j)\in E^a} \frac{1}{\gamma_k} \tau_i^h \pi_{ij}^h s_{ij}(v) + \sum_{i \in E^h} \tau_i^h w_i^h$$

where the subscript $k$ on $\gamma$ refers to the index $k$ for which $h \in P_k$. For the second summation, recall that $s_{ij}(v)$ for all $(i,j) \in E^a$ represents the travel time per automobile and the factor $\frac{1}{\gamma_k}$ converts this time into the same unit as those in the first summation, i.e., time per traveller.

When generalized to the multi-modal case, the Wardrop's first principle reduces to the following conditions for user equilibrium:

$$C_h(\bar{v},\bar{x},\bar{t}) \quad \begin{cases} = \rho_k^1 & \text{if } \bar{x}_h > 0 \\ \geq \rho_k^1 & \text{if } \bar{x}_h = 0 \end{cases}, \forall k, h \in P_k^{am} \cup P_k^{wm}, \tag{1}$$

$$C_h(\bar{v},\bar{x},\bar{t}) \quad \begin{cases} = \rho_k^2 & \text{if } \bar{x}_h > 0 \\ \geq \rho_k^2 & \text{if } \bar{x}_h = 0 \end{cases}, \forall k, h \in P_k^a, \tag{2}$$

$$u_k(\bar{t}_k) \quad = \rho_k^1 - \rho_k^2, \forall k. \tag{3}$$

The first two conditions ensure that the costs of the utilized hyperpaths are less than the cost of those not utilized for both auto and mixed options. Under the binomial logit function, the last condition is equivalent to the following:

$$\bar{t}_k = \frac{b_k}{1 + \exp\{(\lambda^m - \lambda^a) + \eta(\rho_k^1 - \rho_k^2)\}}.$$

In other words, the travel demand under mixed option must equal to the one specified by the binomial logit function. Intuitively, this implies that travellers have no incentive to switch their travel options at equilibrium. The theorem below shows that the above user equilibrium conditions, (1) – (3), reduce to a variational inequality.

**Theorem 2** *A feasible solution $(\bar{v},\bar{x},\bar{t}) \in X$ is in user equilibrium if and only if it solves the following variational inequality (also denoted as VI[$(s(v), W, -u(t)), X$] ):*

$$s(\bar{v})^T(v - \bar{v}) + W^T(x - \bar{x}) - u(\bar{t})^T(t - \bar{t}) \geq 0, \forall (v,x,t) \in X.$$

**Proof:** Let $(\bar{v},\bar{x},\bar{t}) \in X$ satisfies (1) to (3). Observe that (1) implies that $(C_h(\bar{v},\bar{x},\bar{t}) - \rho_k^1) \cdot x_h \geq (C_h(\bar{v},\bar{x},\bar{t}) - \rho_k^1) \cdot \bar{x}_h$ for all feasible $x_h$ and $h \in P_k^{am} \cup P_k^{wm}$ because each term in the first product

is nonnegative and the second product must equal zero. Similarly, (2) also implies that $(C_h(\bar{v},\bar{x},\bar{t}) - \rho_k^2)\cdot x_h \geq (C_h(\bar{v},\bar{x},\bar{t}) - \rho_k^2)\cdot \bar{x}_h$ for all feasible $x_h$ and $h \in P_k^a$. From these two inequalities, we have

$$\begin{aligned}(C_h(\bar{v},\bar{x},\bar{t}) - \rho_k^1)(x_h - \bar{x}_h) &\geq 0, \forall k, h \in P_k^{am} \cup P_k^{wm} \\ (C_h(\bar{v},\bar{x},\bar{t}) - \rho_k^2)(x_h - \bar{x}_h) &\geq 0, \forall k, h \in P_k^{a}\end{aligned}$$

By summing over $h$ and $k$, these inequalities yield the following

$$\begin{aligned}\sum_k \sum_{h\in P_k} C_h(\bar{v},\bar{x},\bar{t})(x_h - \bar{x}_h) &\geq \sum_k \left( \sum_{h\in P_k^{wm}\cup P_k^{am}} \rho_k^1(x_h - \bar{x}_h) + \sum_{h\in P_k^a} \rho_k^2(x_h - \bar{x}_h)\right) \\ &= \sum_k \left( \rho_k^1 \sum_{h\in P_k^{wm}\cup P_k^{am}} (x_h - \bar{x}_h) + \rho_k^2 \sum_{h\in P_k^a} (x_h - \bar{x}_h)\right) \\ &= \sum_k \left(\rho_k^1(t_k - \bar{t}_k) + \rho_k^2([b_k - t_k] - [b_k - \bar{t}_k])\right) \\ &= \sum_k \left(\rho_k^1(t_k - \bar{t}_k) - \rho_k^2(t_k - \bar{t}_k)\right) \\ &= \sum_k (\rho_k^1 - \rho_k^2)(t_k - \bar{t}_k) = u(\bar{t})^T(t - \bar{t})\end{aligned}$$

or

$$\sum_k \sum_{h\in P_k} C_h(\bar{v},\bar{x},\bar{t})(x_h - \bar{x}_h) - u(\bar{t})^T(t - \bar{t}) \geq 0, \forall (v,x,t) \in X. \tag{4}$$

Using the definition of $C_h(\cdot)$, the first term in the above inequality can be written as follows:

$$\begin{aligned}&\sum_k \sum_{h\in P_k} C_h(\bar{v},\bar{x},\bar{t})(x_h - \bar{x}_h) \\ &= \sum_k \sum_{h\in P_k} \left( \sum_{(i,j)\in E^a\cup E^m\cup E^w} \tau_i^h \pi_{ij}^h s_{ij}(\bar{v}) + \sum_{(i,j)\in E^a} \frac{1}{\gamma_k} \tau_i^h \pi_{ij}^h s_{ij}(\bar{v}) + \sum_{i\in E^h} \tau_i^h w_i^h \right)(x_h - \bar{x}_h) \\ &= \sum_{(i,j)\in E^m\cup E^w} \left( s_{ij}(\bar{v}) \sum_k \sum_{h\in P_k} \tau_i^h \pi_{ij}^h (x_h - \bar{x}_h)\right) + \sum_{(i,j)\in E^a} \left( s_{ij}(\bar{v}) \sum_k \frac{1}{\gamma_k} \sum_{h\in P_k} \tau_i^h \pi_{ij}^h (x_h - \bar{x}_h)\right) \\ &\qquad + \sum_k \sum_{h\in P_k} \sum_{i\in E^h} \tau_i^h w_i^h (x_h - \bar{x}_h) \\ &= s(\bar{v})^T(v - \bar{v}) + W^T(x - \bar{x})\end{aligned}$$

Therefore, (4) can be written as

$$s(\bar{v})^T(v - \bar{v}) + W^T(x - \bar{x}) - u(\bar{t})^T(t - \bar{t}) \geq 0, \forall (v,x,t) \in X.$$

Conversely, if $(\bar{v}, \bar{x}, \bar{t}) \in X$ solves the VI, then it must also solves the following linear program (see, e.g., [7] ):

$$\begin{aligned} \min \quad & s(\bar{v})^T v + W^T x - u(\bar{t})^T t \\ \text{s.t.} \quad & v = Z \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ & A_1 x_1 = t \\ & A_2 x_2 = b - t \\ & x_1, x_2 \geq 0. \end{aligned}$$

Then, the optimality conditions of the above linear program reduce to (1) to (3). ■

## 5 Toll Pricing

Let $\beta$ be a vector of tolls. Then, a solution $(v^\beta, x^\beta, t^\beta) \in X$ is in a tolled user equilibrium if the following holds:

$$(s(v^\beta) + \beta)^T (v - v^\beta) + W^T (x - x^\beta) - u(\bar{t})^T (t - t^\beta) \geq 0, \forall (v, x, t) \in X$$

In other words, $(v^\beta, x^\beta, t^\beta)$ solves VI$[(s(v) + \beta, W, -u(t)), X]$. As in [12], we say that $\beta$ is a valid toll vector if $(v^\beta, x^\beta, t^\beta)$ also solves the system problem. When $s(v)$ is strictly monotone and $s(v)^T v$ is strictly convex, the set of valid toll vectors or the toll set is

$$\mathcal{T}(v^*, x^*, t^*) = \{\beta : (v^\beta, x^\beta, t^\beta) = (v^*, x^*, t^*)\}$$

The theorem below provides an algebraic expression for the above toll set.

**Theorem 3** *The toll set, $\mathcal{T}(v^*, x^*, t^*)$, consists of the $\beta$ component of every vector $(\beta, \rho)$ that satisfies the following linear system:*

$$\begin{aligned} Z^T (s(v^*) + \beta) + W &\geq \begin{bmatrix} A_1^T \rho^1 \\ A_2^T \rho^2 \end{bmatrix} \\ u(t^*) &= \rho^1 - \rho^2 \\ [s(v^*) + \beta]^T v^* + W^T x^* &= (t^*)^T \rho^1 + (b - t^*)^T \rho^2 \end{aligned}$$

**Proof:** The result follows from Proposition 1.2.1 in [7] and linear programming duality. ■

Implicit in the above theorem is the assumption that every arc in the mixed networks is tollable. However, the walking arcs, i.e., $E^w$, are not tollable. In some applications, it may not be intuitive to toll embarking and disembarking arcs. Thus, the following restricted toll set may be more practical:

$$\mathcal{T}_0(v^*, x^*, t^*) = \{\beta : (v^\beta, x^\beta, t^\beta) = (v^*, x^*, t^*), \beta_{ij} = 0, \forall (i,j) \in E^w \cup E^{ed}\}$$

where $E^{ed}$ is the set of embarking and disembarking arcs and $E^{ed} \subseteq E^m$. The following result provides conditions under which the above toll set is non-empty. Without this result, the toll pricing problems in the next section must be formulated as mathematical programs with equilibrium constraints (see, e.g., [16]), a class of problems that is more difficult to solve.

**Corollary 1** *If $s_{ij}(v_{ij}) = c_{ij}$ for all $(i,j) \in E^w \cup E^{ed}$, where $c_{ij}$ is a constant, then the restricted toll set, $\mathcal{T}_0(v^*, x^*, t^*)$, is non-empty and consists of the $\beta$ component of every vector $\beta$, $\rho$ that satisfies the following linear system:*

$$\begin{aligned}
Z^T(s(v^*) + \beta) + W &\geq \begin{bmatrix} A_1^T \rho^1 \\ A_2^T \rho^2 \end{bmatrix} \\
u(t^*) &= \rho^1 - \rho^2 \\
[s(v^*) + \beta]^T v^* + W^T x^* &= (t^*)^T \rho^1 + (b - t^*)^T \rho^2 \\
\beta_{ij} &= 0, \qquad \forall (i,j) \in E^w \cup E^{ed}
\end{aligned}$$

**Proof:** The algebraic expression for the restricted toll set, $\mathcal{T}_0(v^*, x^*, t^*)$, follows from Theorem 3. Moreover, Theorem 1 demonstrates that the marginal social cost pricing (MSCP) vector, $\nabla s(v^*)^T v^*$, is valid, i.e., it satisfies the first three conditions in the above system. However, under the hypothesis, the marginal social costs for arcs in the set $E^w \cup E^{ed}$ are zero, i.e., $\left[\nabla s(v^*)^T v^*\right]_{ij} = 0$ for all $(i,j) \in E^w \cup E^{ed}$. Thus, $\nabla s(v^*)^T v^*$ satisfies the above linear system, i.e., $\mathcal{T}_0(v^*, x^*, t^*)$ is non-empty. ■

Similarly, the following corollary provides a sufficient condition under which the restricted and nonnegative toll set, i.e.,

$$\mathcal{T}_0^+(v^*, x^*, t^*) = \{\beta \geq 0 : (v^\beta, x^\beta, t^\beta) = (v^*, x^*, t^*), \beta_{ij} = 0, \forall (i,j) \in E^w \cup E^{ed}\}$$

is non-empty.

**Corollary 2** *If $\nabla s(v^*)^T v^*$ is a nonnegative vector, then $\mathcal{T}_0^+(v^*, x^*, t^*)$ is non-empty and consists of the $\beta$ component of every vector $(\beta, \rho)$ that satisfies the following linear system:*

$$\begin{aligned}
Z^T(s(v^*) + \beta) + W &\geq \begin{bmatrix} A_1^T \rho^1 \\ A_2^T \rho^2 \end{bmatrix} \\
u(t^*) &= \rho^1 - \rho^2 \\
[s(v^*) + \beta]^T v^* + W^T x^* &= (t^*)^T \rho^1 + (b - t^*)^T \rho^2 \\
\beta_{ij} &= 0, \qquad \forall (i,j) \in E^w \cup E^{ed} \\
\beta_{ij} &\geq 0, \qquad \forall (i,j) \notin E^w \cup E^{ed}
\end{aligned}$$

**Proof:** The result follows immediately from the hypothesis and Corollary 1. ■

In general, the number of valid toll vectors is infinite. Below are models for choosing a valid toll vectors, some of which are motivated by the models in [12].

1. **Minimum Financial Burden:** This problem minimizes the revenue from automobile tolls and transit fare increases (i.e., tolls on the metro system) and can be stated as follows:

$$\begin{aligned}
\text{MINREV1:} \qquad & \min_{(\beta,\rho)} \ \beta^T v \\
& \text{s.t.} \quad (\beta, \rho) \in \mathcal{T}_0^+(v^*, x^*, t^*)
\end{aligned}$$

2. **Minimum Toll Revenue with Transit Subsidy:** This problem charges tolls on the auto network and uses them to subsidize the fare discounts on the metro system. The objective of this problem is to minimize the toll revenue while ensuring that it is sufficient to subsidize the fare discounts.

$$\begin{aligned}
\text{MINREV2:} \qquad & \min_{(\beta,\rho)} \ \beta^T v \\
& \text{s.t.} \quad (\beta, \rho) \in \mathcal{T}_0(v^*, x^*, t^*) \\
& \qquad \sum_{(i,j) \in E^m - E^{ed}} -\beta_{ij} v \leq \sum_{(i,j) \in E^a} \beta_{ij} v, \\
& \qquad -s_{ij}(v_{ij}) \leq \beta_{ij} \leq 0, \qquad \forall (i,j) \in E^m - E^{ed} \\
& \qquad \beta_{ij} \geq 0, \qquad \forall (i,j) \in E^a
\end{aligned}$$

3. **Minimum Toll Booth on the Auto Network:** This problem minimizes the number of toll booths required for toll collection on the auto network. There is no need to minimize the number of toll booths on the metro part of the mixed networks because we interpret a toll on a metro

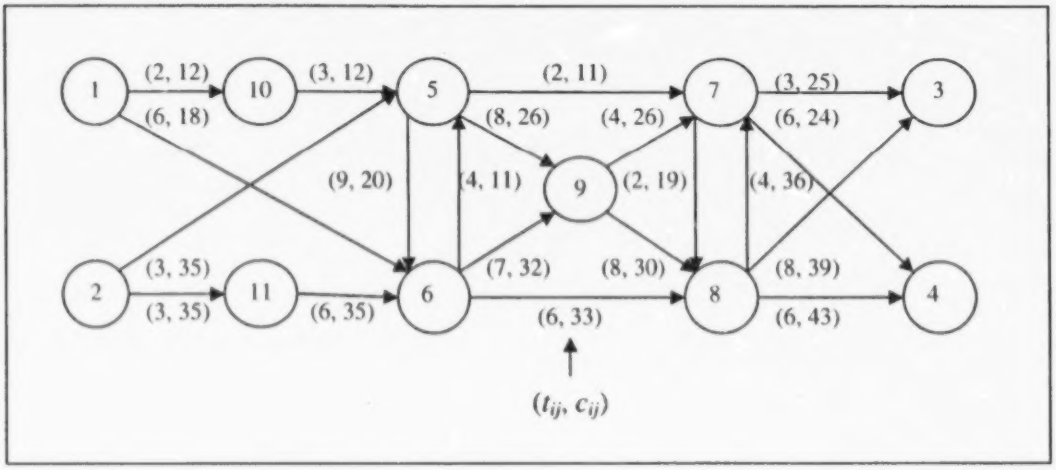



Figure 7: Auto network

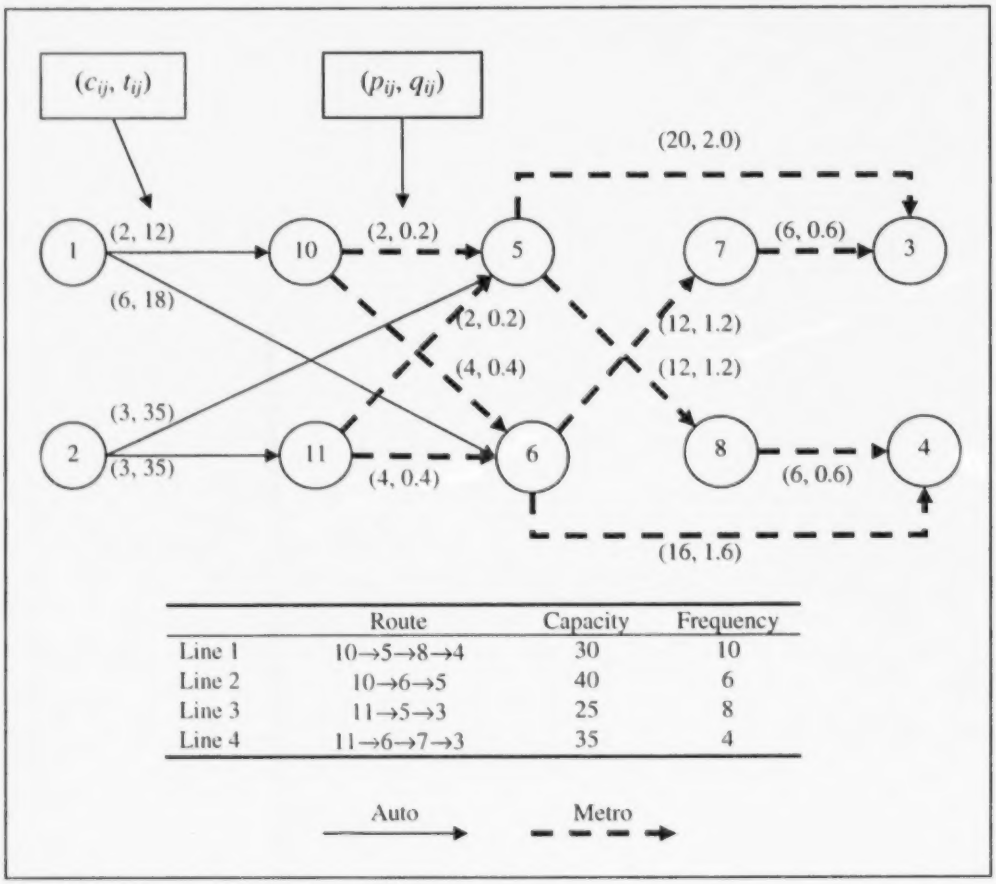
arc as a fare increase.

$$
\begin{aligned}
\text{MINTB1:} \quad & \min_{(\beta,\rho)} \quad \sum_{(i,j)\in E^a} y_{ij} \\
& \text{s.t.} \quad (\beta,\rho) \in \mathcal{T}_0^+(v^*, x^*, t^*) \\
& \qquad \beta_{ij} \le M y_{ij}, \qquad \forall (i,j) \in E^a \\
& \qquad y_{ij} \in \{0,1\}, \qquad \forall (i,j) \in E^a
\end{aligned}
$$

where $M$ is a sufficiently large constant.

4. **Minimum Toll Booth with Transit Subsidy:** This is similar to the minimum toll revenue problem with transit subsidy.

$$
\begin{aligned}
\text{MINTB2:} \quad & \min_{(\beta,\rho)} \quad \sum_{(i,j)\in E^a} y_{ij} \\
& \text{s.t.} \quad (\beta,\rho) \in T_0(v^*, x^*, t^*) \\
& \qquad 0 \le \beta_{ij} \le M y_{ij}, \qquad \forall (i,j) \in E^a \\
& \qquad y_{ij} \in \{0,1\}, \qquad \forall (i,j) \in E^a \\
& \qquad \sum_{(i,j)\in E^m - E^{ed}} -\beta_{ij} v \le \sum_{(i,j)\in E^a} \beta_{ij} v, \\
& \qquad -s_{ij}(v_{ij}) \le \beta_{ij} \le 0, \qquad \forall (i,j) \in E^m - E^{ed}
\end{aligned}
$$




| | Route | Capacity | Frequency |
|---|---|---|---|
| Line 1 | 10→5→8→4 | 30 | 10 |
| Line 2 | 10→6→5 | 40 | 6 |
| Line 3 | 11→5→3 | 25 | 8 |
| Line 4 | 11→6→7→3 | 35 | 4 |

Figure 8: Consolidated auto-metro network

## 6 Numerical Example

To illustrate the differences in the four toll pricing problems in the previous section, consider the auto network in Figure 7. The travel cost functions for arcs in this network (i.e., $(i, j) \in E^a$) are of the form $s_{ij}(v_{ij}) = t_{ij}(1 + 0.15(v_{ij}/c_{ij})^4)$, where $t_{ij}$ represents the free-flow travel time and $c_{ij}$ is the arc capacity. In the figure, the pair $(t_{ij}, c_{ij})$ are shown next to each arc.

To simplify our presentation, we assume that there is no walking. As a result, there is only one mixed network, the auto-metro network. Figure 8 displays the consolidated version (i.e., the transfer points and embarking and disembarking arcs are not shown) of the auto-metro network for our example. Arcs (1, 10), (1,6), (2, 5) and (2, 11) in this auto-metro network are the same as those in Figure 7, i.e., they are auto arcs. The remaining arcs (or metro arcs) correspond to segments of four

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| H1 | [10] | ∅ | ∅ | ∅ | [3] | [7] | [3] | ∅ | ∅ | [5,6] | ∅ |
| H2 | [10] | ∅ | ∅ | ∅ | ∅ | [7] | [3] | ∅ | ∅ | [6] | ∅ |
| H3 | [10] | ∅ | ∅ | ∅ | [3] | ∅ | ∅ | ∅ | ∅ | [5] | ∅ |
| H4 | [6] | ∅ | ∅ | ∅ | ∅ | [7] | [3] | ∅ | ∅ | ∅ | ∅ |

Table 2: Hyperpaths for OD pair (1,3)

metro lines listed in the table underneath the network. Similar to the auto network, the travel cost functions for the metro arcs are of the form $s_{ij}(v_{ij}) = p_{ij} + q_{ij}(v_{ij}/c_{ij})^2$. The first term, $p_{ij}$, in this expression represents the fare (in units of time) for the $(i,j)$ segment of the transit line. The second term represents the actual travel time that depends on a constant $q_{ij}$ and $c_{ij}$ which, as before, is the arc capacity. In Figure 8, the pair $(p_{ij}, q_{ij})$ are shown next to each arc. Each segment $(i,j)$ of the same metro line has the same capacity and frequency and they are listed in the table underneath the network.

For each OD pair, there are four possible hyperpaths. For example, Table 2 lists those for OD (1,3). Each of the last eleven columns represents a node and the numbers listed in each one represent the set of attractive successors, i.e., $E^h(i^+)$, associated with the node in that column. For example, the first row in the table lists the sets of attractive successors for hyperpath $H1$. For this hyperpath, the sets of attractive successors at nodes 1, 5, 6, 7 and 10 are $\{10\}$, $\{3\}$, $\{7\}$, $\{3\}$ and $\{5,6\}$, respectively. The sets of attractive successors for the remaining nodes are empty. Moreover, hyperpath $H1$ contains two simple paths $1 \rightarrow 10 \rightarrow 5 \rightarrow 3$ and $1 \rightarrow 10 \rightarrow 6 \rightarrow 7 \rightarrow 3$.

Finally, there are four OD-pairs: $(1,3)$, $(1,4)$, $(2,3)$ and $(2,4)$ with demands $o_{(1,3)} = 10$, $b_{(1,4)} = 20$, $b_{(2,3)} = 30$ and $b_{(2,4)} = 40$, respectively. The occupancy rate for all OD pair is one. For the binomial logit function, we set $\eta = 0.04$ and $(\lambda^m - \lambda^a) = ln(3)$.

Table 3 displays the shares of travel demands for the auto and auto-metro options from the user and system solutions. Of interest is the share of demand for the auto-metro option because it represents the transit ridership. Of the 100 travellers, the system solution has 3.6948 (or 3.69%) more travellers using the metro than the user solution. However, when compared to the transit ridership (19.6359) in the user solution, 3.6948 travellers represents nearly a 19% increase in transit ridership. One method for achieving this increase is via congestion pricing.

| OD pair | Mode | User Eq. | Sys. Opt. | Change (%) |
|---|---|---|---|---|
| (1,3) | Auto | 8.5046 | 8.1925 | -3.12% |
| | Auto-Metro | 1.4954 | 1.8075 | 3.12% |
| (1,4) | Auto | 15.8361 | 14.8817 | -4.77% |
| | Auto-Metro | 4.1639 | 5.1183 | 4.77% |
| (2,3) | Auto | 24.3958 | 22.7466 | -5.50% |
| | Auto-Metro | 5.6042 | 7.2534 | 5.50% |
| (2,4) | Auto | 31.6276 | 30.8485 | -1.95% |
| | Auto-Metro | 8.3724 | 9.1515 | 1.95% |
| Total | Auto | 80.3641 | 76.6693 | -3.69% |
| | Auto-Metro | 19.6359 | 23.3307 | 3.69% |

Table 3: Shares of demands for the auto and auto-metro options

Table 4 offers five congestion pricing alternatives. One is from marginal social cost pricing and the remaining four are from solving the problems in Section 5. Observe that MINREV1 does not toll any arc on the metro network. As expected, the toll revenue from MINREV1 is less than that of MINTB1. For the cases with subsidy, the similar also holds for MINREV2 and MINTB2. (Note that the fact that MINREV2 and MINTB2 use the same number [four] of tollbooths is coincidental. In general, we expect MINREV2 to use more.) Naturally, the revenues for the case without subsidy (MINREV1 and MINTB1) are more than those with subsidy (MINREV2 and MINTB2). For the latter, the toll revenue from automobiles are also more than sufficient to finance the subsidy given to transit users in terms of reduced ticket prices. Perhaps, the excess revenue can be used to improve the transit systems such as adding new metro lines or increasing the frequency of each line, e.g., by adding more trains to the metro system.

To examine the effect of not fully implementing congestion tolls, we solved the tolled user equilibrium problem, $\mathrm{VI}[(s(v)+\beta, W, -u(t)), X]$ with $\beta = \alpha \nabla s(v^*)^T v^*$ where $v^*$ solves the system problem and $\alpha \in [0,1]$. (With minor round-off errors, the results using tolls from MINREV1, MINTB1, MINREV2, and MINTB2 are the same.) Along with other information, Table 5 displays the resulting number of travellers using the auto-metro option from solving the tolled user equilibrium using eleven different values of $\alpha$. When $\alpha = 0$, the tolled user equilibrium problem reduces to the user problem

| | $\nabla s(v)^T v$ | MINREV1 | MINTB1 | MINREV2 | MINTB2 |
|---|---|---|---|---|---|
| **Auto arcs** | | | | | |
| (1,6) | 3.09 | 2.90 | | | |
| (1,10) | 1.50 | | | | |
| (2,5) | 3.70 | 3.26 | 4.00 | 4.00 | 4.00 |
| (2,11) | 0.75 | | | | |
| (5,7) | 10.88 | 10.59 | 11.43 | 8.00 | 8.00 |
| (5,9) | 1.50 | 2.59 | | | |
| (6,8) | 4.51 | | 7.43 | | 4.00 |
| (6,9) | 0.00 | | | | |
| (7,3) | 3.12 | 2.90 | 4.00 | | 4.00 |
| (7,4) | 0.32 | | | | |
| (8,3) | 0.00 | 2.90 | | | |
| (8,4) | 1.20 | 4.00 | | 4.00 | |
| (8,7) | | 0.22 | | 0.22 | |
| (9,7) | 1.39 | | 3.43 | | |
| (10,5) | 0.10 | 0.31 | | | |
| (11,6) | 0.45 | 1.85 | | | |
| **Metro arcs** | | | | | |
| (5,3) | 0.28 | | 1.62 | -3.75 | -0.90 |
| (5,8) | 0.41 | | | | |
| (6,4) | 0.01 | | | | |
| (6,7) | 0.01 | | 4.77 | -2.42 | |
| (7,3) | 0.01 | | | -2.42 | |
| (8,4) | 0.20 | | 0.84 | -2.59 | -2.59 |
| (10,5) | 0.01 | | | -2.06 | -0.90 |
| (10,6) | 0.00 | | | | -1.92 |
| (11,5) | 0.02 | | | | -0.23 |
| (11,6) | 0.00 | | | -1.93 | |
| Auto Rev. | 816.92 | 700.35 | 837.49 | 450.99 | 574.75 |
| Metro Rev. | 9.61 | | 32.64 | -83.55 | -47.31 |
| Total Rev. | 826.53 | 700.35 | 870.13 | 367.44 | 527.44 |

Table 4: Alternative tolls (arcs with no toll are not listed)

| $\alpha$ | Auto-Metro Users | Increase | Auto Users | System Delay |
|---|---|---|---|---|
| 0 | 19.64 | 0.00% | 80.36 | 2269.04 |
| 0.1 | 19.98 | 1.75% | 80.02 | 2249.89 |
| 0.2 | 20.33 | 3.53% | 79.67 | 2231.90 |
| 0.3 | 20.69 | 5.35% | 79.31 | 2215.13 |
| 0.4 | 21.05 | 7.21% | 78.95 | 2199.70 |
| 0.5 | 21.42 | 9.11% | 78.58 | 2185.78 |
| 0.6 | 21.81 | 11.05% | 78.20 | 2173.58 |
| 0.7 | 22.19 | 13.03% | 77.81 | 2163.42 |
| 0.8 | 22.59 | 15.07% | 77.41 | 2155.69 |
| 0.9 | 23.01 | 17.17% | 76.99 | 2150.97 |
| 1 | 23.33 | 18.82% | 76.67 | 2150.75 |

Table 5: Partial Implementation of MSCP tolls

whose solution has 19.64 travellers using the auto-metro option. Similarly, when $\alpha = 1$, the tolled user equilibrium problem becomes the system problem whose solution has 23.33 auto-metro users. When compared with the number from the user solution (i.e., $\alpha = 0$), this represents an 18.82% increase in transit ridership. The last two columns in Table 5 provide the corresponding number of auto users and the associated system delay, i.e., the total travel and waiting time. The latter demonstrates the benefit of congestion pricing–the reduced system delay.

## 7 Conclusion

This paper presents two models for multi-modal transportation systems, user equilibrium and system optimum. The objective of the system problem is to maximize the partial net user benefit and the user equilibrium conditions are equivalent to a variational inequality. In literature, it is often assumed that travellers use simple paths to reach their destinations. To make our models more realistic, we assume instead that travellers adopt strategies or hyperpaths when using the transit system.

From the system solution, we construct the toll set or the set of valid congestion toll vectors and provide a sufficient condition under which tolls on the walk, embarking, and disembarking arcs can

be set to zero. We also propose four toll pricing problems suitable for the multi-modal setting and demonstrate the potential for their use in practice via a small numerical example.

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